Cognitive structures and students' understanding of mathematics

Susanne Bellmer

Department of Informatik, Ostfalia University of Applied Sciences Wolfenbüttel, Germany

Abstract

Learning is a complex process. It does not mean just writing on an empty blackboard or substituting old knowledge by new information, but it is connected with cognitive structures. During the learning process cognitive structures are built and new information is integrated within already existing manifold structures. These structures have a strong influence, if and how information is understood.
Up to now most of the existing studies have dealt with physics. But are these structures also important in mathematics? Investigations reveal that these structures do of course exist in mathematics, and they have a very noticeable effect.

In this contribution the application of the concept of cognitive structures to mathematics is presented and various examples are used to show their effects. It is also proposed how this can be taken into account to develop an effective teaching, so that the learning process is facilitated and persistent mistakes can easier be removed.

Introduction

Year by year lecturers are confronted with the situation, that at any time different students have different levels of understanding and that their knowledge grow with different tempos. Moreover, typical mistakes occur and are quite resistant even to good teaching. The reason is that learning does not mean to write on an empty blackboard or to substitute old knowledge by new information. Learning is a process, which starts at the already existing knowledge and develops gradually, until the new information is understood and therefore new knowledge is gained.

Investigations show, that all existing knowledge and the information connected with it form cognitive structures. These structures have a strong influence on the learning process and the understanding of information. Therefore they must be taken into account to develop an effective teaching.

The following contribution explains the properties of cognitive structures and their influence on the understanding of mathematics. In a second part examples for cognitive structures and possible ways of successfully dealing with them are presented.

What are cognitive structures?

All knowledge we have is not retained in our mind as isolated information pieces, but cognitive structures are built. These structures are connected to the context, in which they are formed, so that associations and logical connections to other subjects exist, even to those completely different from the actual topic. During the learning process new information is integrated into the existing structures and these structures are extended. The effect of cognitive structures depends on the context, in which they are formed first, on the time at which this took place and on their properties as strength, relation to other structures etc. Niedderer (1996) and Niedderer and Petri (2001) have discussed the effect of cognitive structures in the context of physics. But cognitive structures are also relevant in mathematics. A lot of experience exists already in statistics. Konold (1995) discusses some typical mistakes and the related misconceptions. Prediger (2008a) investigates in detail the prevalent misconceptions of probability, the resulting mistakes and the connection with cognitive structures. The situation in calculus is different. Up to now the knowledge of cognitive structures in calculus, their properties and effects is much less than in statistics. Prediger investigated the effect of some structures on fractions (2004) and on calculus (2008b). But there is great need of more expertise and experience here.

Therefore the following contribution discusses the effects of several different cognitive structures on mathematics, especially calculus.

The contexts, in which the cognitive structures relevant for mathematics are formed, are
- mathematics at school or university
- science of nature (physics, chemistry etc.), engineering, computer science
- language and the arts
- everyday life

The times, at which structures are formed, are quite different. In the following they are divided into three time segments:
- very early time, primary school
- time of middle and late school years
- not long ago, already at university

General properties of the structures are for example strength, degree of activation, i.e. always activated because of fundamental character or just sometimes, and the possibility to extend them, i.e. the structures are very rigid or relatively easy to extend.

When the knowledge and principles connected with the already existing structures can successfully be applied in the context of new information and new problems, the problems can be solved correctly and the structure is extended and its strength increases. But when the application is not possible, the understanding is hindered and mistakes occur. These mistakes are typical for the special conflict and difficult to remove.

The longer these structures have already existed, the more connections to other topics and their structures are formed and the greater their influence is. Therefore most of the problems in mathematics are found in calculus (e.g. numbers, functions) and statistics and calculus of probability. Of course these effects can also be found in other subjects of mathematics, but because those are new for the students at university, there are less conflicts with old cognitive structures.

Teaching and cognitive structures

The strong influence of cognitive structures on the learning process shows, that being informed about their existence and their properties is an important basis for good teaching.

First the existing structures have to be identified. How is this possible? The learning history gives important information concerning the already existing knowledge of the considered subject, knowledge of related subjects and the attended school or university. Very important indicators are typical mistakes or wrong concepts. They become apparent in calculations, explanations or argueing. Special phrases or sentences used by the students are a rich source of hints.

Next the properties of the structures should be investigated. One should know: Which knowledge is retained there? Where does it come from? In which area do they yield correct results? Where is the boundary? Where are conflicts between old and actually taught information? Is it a strong structure and therefore easily activated? How is the structure activated, i.e. is the context most important or do special words act as signals? Which typical mistakes can occur?

After these investigations the existing cognitive structures can be taken into account, and a more effective teaching can be developed. Usually this is possible on two ways: First the expansion of the existing structures can be facilitated by using explanations and exercises, which start at the existing structures and lead step by step to the new knowledge. Second it can be demonstrated that the knowledge and methods connected with the old structures can not be applied in the new context and therefore new information and methods are necessary.

Examples for Dealing with Cognitive Structures

In the following six different cognitive structures are discussed as concrete examples. Every structure is characterized by the time it is formed and the context.
1) Old structure, built until the age of 10, context of calculus
2) Structure formed between the age of 12 and the age of 18, context of calculus
3) Very old structure, formed during the whole life, context of everyday life
4) Structure formed until the age of 14 or in the present, context of physics
5) Expanding an existing structure, context of calculus

1) Old structure, built until the age of 10, context of calculus

a) When students have to determine the poles of a function or when they have to use l'Hospitals rule, mistakes like $\frac{6}{0}=0$ or $\frac{6}{0}=6$ occur.

This problem has several reasons. When division or when calculations using the number zero are taught, explanations may be used, which are applicable to divisions like $\frac{6}{2}$, but not to $\frac{6}{0}$.
Sentences used in the context of divisions can result in both mistakes mentioned above. The first explanation usually applied to the division by zero is "6 EUR are distributed among 0 people, so nobody gets anything." This causes the mistake $\frac{6}{0}=0$. Another explanation results in the rationale "6 EUR are distributed among 0 people, so there are still 6 EUR afterwards." and therefore in the mistake $\frac{6}{0}=6$.
As soon as students are able to describe a division in a way applicable to $\frac{6}{0}$, they get the correct result and can give a correct explanation.
The mistake $\frac{6}{0}=6$ also has its origin in an inadequate use of a rule taught in primary school. When calculations using the number zero are taught, the pupils learn how to add or subtract zero.

In this context the sentence "Zero means to do nothing" may be used (Didaktischer Informationsdienst Mathematik 1987). An application to division yields the wrong result $\frac{6}{0} = 6$.

The usual way to deal with this problem is discussing fractions, whose numerator is constant and whose denominator decreases, for example:

$\frac{6}{1/10}, \frac{6}{1/100}, \frac{6}{1/1000}, \ldots, \frac{6}{0}$.

This explanation is understood by the students, but it does not really help, because the existing cognitive structure is not taken into account.
A way out of the dilemma is the usage of the following adjusted explanations:
$\frac{6}{2} = 3$ - How often do I have to add 2 to get 6? How often can I give 2 people 1 EUR until 6 EUR are spent? Three times.
$\frac{6}{0}$ - How often do I have to add 0 to get 6? How often can I give 1 person 0 EUR until 6 EUR are spent? Infinitely often.
After these explanations even weak students have responded: "Oh, that is logical!"

b) Decimal numbers are used in many subjects, e.g. calculus, numerical methods or computer science. Again and again students state that the number 0.251421 is greater than 0.26.

This mistake has its root in primary school. When natural numbers are learnt, the phrase "the more digits, the greater the number" may be formed. While this is true for natural numbers, it does not work for other numbers, especially for rational numbers.

In this case confronting the wrong sentence and giving a few explanations is already effective.

2) Structure formed between the age of 12 and the age of 18, context of calculus

A very common problem concerns functions (Bellmer, 2009). Students do not define a function to be a special relation, but they consider a function to be equivalent to its representations equation or graph. This problem does only occur in calculus and not in algebra, where the students use the different representations of a relation very well and without confusing the relation with its representation.

This can be explained as follows: When functions are taught at school, they are defined as unique relations. But this is mentioned nearly never again afterwards, and all the time calculations are carried out or graphs are plotted. Therefore over the years the original definition is forgotten, and the students think, that a function is equivalent to an equation or a graph. Moreover the graph is thought to be a rigid object, and its construction by using the equation to determine the coordinates of different points is completely forgotten.

A hard work has to be done to solve this problem, because the structure is very strong and rigid. The concept of a relation must be taught again and every representation of a function has to be discussed in this context very often. A lot of special teaching and exercises are necessary here. But finally there are positive results.

3) Very old structure, formed during the whole life, context of everyday life

The meaning of the derivative of a function being the rate of change at one point is not understood by a significant percentage of the students.

This misunderstanding is caused by a conflict with the situation in everyday life. In everyday life the change of the temperature or of the amount of money on a bank account is determined without mentioning the time interval in which this change takes place. So the change of temperature is given in °C or the change of money in EUR. When students apply this to mathematics, they consider only the values of the function itself to determine the rate of change and do not use the derivative.

An effective way out is teaching the difference quotient first and discussing it using suitable examples and exercises and then leading the students step by step to the concept of the differential quotient. The graphical differentiation of given graphs is also a very good exercise. A training like this is very effective and sustained success can be achieved.

Since the last few years this is the usual way at german schools, and the pupils' understanding of the derivative as the rate of change at one point has become much better and hence they are much better prepared for university.

4) Formed at the age of about 14 or in the present, context of physics

Some students confuse the terms "concave" and "convex" curvature of a function.

This confusion has its origin in a conflict with a mnemonic tric used in physics. When the different types of lenses are introduced, the mnemonic tric "One can pour coffee into a concave lens" (German: "In eine konkave Linse kann man Kaffee gießen") is used. The strength of this structure depends on the person's knowledge of physics, especially optics.

New mnemonic trics, which should help to identify the curvature correctly, only work well, when there is no interaction with the mentioned cognitive structure.
A way out is an adjusted mnemonic tric which uses the already existing one: "In math the coffee is poured in from below." (German: "In der Mathematik kommt der Kaffee unten 'rein.") This funny sentence is very effective.

5) Expanding an existing structure, context of calculus

When a function f, its first derivative f' and its second derivative f" are discussed, students find it difficult to recognize, on which level the considered term or graph has to be interpreted.

This difficulty arises, because students are used to deal with only one function. Therefore they interpret every graph or term presented as a representation or description of one function, the function f.

Very good exercises to train switching between different levels of interpretation are:

a) Given the graph of f (or f') the corresponding graph of f' (or f) has to be constructed or has to be identified among a number of graphs. The construction or choice has to be explained in detail.
b) One single graph is presented three times, and the graph represents first f(x) as well as second f'(x) and third f''(x). Then a special point x_0 is marked. At best the signs of f, f' and f'' are different, i.e. $f(x_0)>0$, $f'(x_0)<0$ and $f''(x_0)>0$ (or vice versa in every case). Then the following questions are asked:

- What is the sign of f?
- What is the monotone behaviour of f?
- Is the slope of f positive or negative at the marked point?
- Is $f'(x_0) >0$ true or $f'(x_0) <0$?
- Does the slope increase or decrease at x_0?
- Is $f''(x_0) >0$ true or $f''(x_0) <0$?
- What is the curvature of f(x)? Is f(x) convex or concave?

These exercises are very effective. Even a very good student sighed: "Oh, my head!"

Conclusions

Cognitive structures have a strong influence on the learning process in mathematics and the understanding of new information. Therefore further investigations are planned to identify cognitive structures, so that they can be taken into account and an effective teaching can be developed. As a result the students learning is facilitated and persistent mistakes can easier be removed.

References

Bellmer, S. (2009) "Typical Mistakes Students Make in Mathematics", Talk at *Workshop "Mathematical Education of First-Year Engineering Students"*, TU Berlin, 6.-7.April 2009

Didaktischer Informationsdienst Mathematik "Arithmetik, Algebra und Funktionen" (1987), Curriculum Heft 44, Landesinstitut für Schule und Weiterbildung, ed., Soest

Konold, C. (1995) "Issues in Assessing Conceptual understanding in Probability and Statistics" In *Journal of Statistics Education* V.3, Number 1

Niedderer, H. (1996) "Überblick über Lernstudien in Physik" In Duit, R., and Rhöneck, C. eds. *Lernen in den Naturwissenschaften*, Institut für Pädagogik der Naturwissenschaften an der Universität Kiel, 119-144

Petri, J. and Niedderer, H. (2001) „Kognitive Schichtenstrukturen nach einer UE Atomphysik (S II)" In *Zeitschrift für Didaktik der Naturwissenschaft 7,* 53-68

Prediger, S. (2004) "Brüche bei Brüchen – aufgreifen oder umschiffen?" In *Mathematik lehren* Heft 123 (April 2004), 10-13

Prediger, S. (2008a) „Do you want me to do it with probability or with my normal thinking? – Horizontal and vertical views on the formation of stochastic conceptions" in *International Electronic Journal of Mathematic Education,* Vol. 3, Number 3

Prediger, S. (2008b) "Bestand und Änderung - Ein Beitrag zur didaktischen Rekonstruktion der Analysis" In *Journal für Mathematikdidaktik 29 (2008),* Teubner-Verlag, 163-198.